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The boundary layer in crossed-fields m.h.d.

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This study of the boundary layer of steady, incompressible, plane, crossed-fields m.h.d. flow at large Reynolds number Re and magnetic Reynolds number Rm begins with a review of Hartmann's case, where a boundary layer occurs whose thickness is proportional to $(Re\,Rm)^{-\frac{1}{2}}$. Following this clue, it is shown that in general the boundary layer is a 'local Hartmann boundary layer'. Its profiles are always exponential and it is determined completely by local quantities. The skin friction and the total electric current in the layer are proportional to the square root of the magnetic Prandtl number, i.e. to $(Rm/Re)^{\frac{1}{2}}$. Thus the exterior-flow problem, the solution of which precedes a boundary-layer solution, generally involves a current sheet at the fluid–solid interface.

This inviscid-flow problem becomes tractable if $(Rm/Re)^{\frac{1}{2}}$ is small enough to permit a linearized solution. The flow field about a flat plate at zero incidence is calculated in this approximation. It is pointed out that the thin-cylinder solutions of Sears & Resler (1959), which pertain to $Rm/Re = 0$, can immediately be extended to small, non-zero values of this parameter by linear combination with this flat-plate solution.

1. Introduction

The term 'crossed-fields' has been used to describe certain flows of electrically conducting fluids involving, in some undisturbed region, a uniform fluid stream and a uniform magnetic field making a non-zero angle with it. It is clear that there must be an electric field in such a region, directed perpendicularly to both the stream velocity and magnetic-field vectors. In most investigations (see e.g. Sears & Resler 1959, 1964; Dix 1963; Clauser 1963) further simplification has been achieved by assuming steady conditions and two-dimensional flow.

In some of these investigations, viscosity, electrical resistance, or both, have been neglected. The most significant phenomena studied are then the standing Alfvén waves or (in compressible fluids) magneto-acoustic waves that are produced by the streaming of the fluid past an obstacle or other disturbance. When the diffusive phenomena are included, it is found that the standing waves are diffused, and the resulting regions of vorticity and current, extending parabolically away from the source of disturbance, are sometimes called 'wakes'.

Clauser (1963) undertook to systematize crossed-fields flow by studying the 'field modes' exhibited by the partial-differential equation system of small perturbations. One of his conclusions is that 'we can never have boundary layers or wakes in the conventional sense'. In particular, he finds, when the two diffusive parameters are equal ('magnetic Prandtl number' equal to one), that 'no boundary layer or wake phenomena can possibly exist'. (Here of course, he is using the word 'wake' correctly, to mean only the vortical-flow region directly downstream of a disturbance.)

These conclusions of Clauser are somewhat puzzling in the light of several investigations where phenomena are revealed that surely resemble *boundary layers*; i.e. they are thin regions (for small viscosity and resistivity) adjacent to a solid–fluid interface, in which the tangential component of velocity varies rapidly and there is large vorticity. The results of both Dix (1963) and Bryson & Rosciszewski (1962), involving crossed-fields flow past a flat plate, exhibit such behaviour unmistakably. There is also an earlier study by Stewartson (1960) which the present author criticized (Sears 1961) in its application to aligned-fields flow, but which surely exhibits in crossed-fields flow a kind of boundary layer adjacent to a plane wall for arbitrary non-zero values of the diffusion parameters. Actually, the essence of Stewartson's analysis was anticipated by Hartmann in his classical study (1937) of magnetohydrodynamic Poiseuille flow. For small viscosity and/or resistance the velocity profile is flat over most of the channel width, and there is a boundary layer at each wall.

It is therefore of some interest to examine the subject of crossed-fields boundary layers more closely and to reconcile, if possible, Clauser's sweeping statements with the indications of other investigations.

2. The boundary layer of Hartmann flow

We begin by reviewing briefly the classical case studied by Hartmann; i.e. flow of an incompressible conducting fluid between parallel plane walls in the presence of perpendicular transverse magnetic and electric fields and a uniform pressure gradient.

The equations for incompressible steady flow are, in e.m.u. (see Sears & Resler 1964),

$$\mathbf{v}.\nabla\mathbf{v}-\frac{\mu}{4\pi\rho}\mathbf{H}.\nabla\mathbf{H}+\frac{1}{\rho}\operatorname{grad}\left\{p+\frac{\mu\mathbf{H}^2}{8\pi}\right\}=\nu\nabla^2\mathbf{v}, \tag{1}$$

and

$$\operatorname{curl}\mathbf{H}=4\pi\sigma\{\mathbf{E}+\mu\mathbf{v}\times\mathbf{H}\}, \tag{2}$$

together with the conditions $\operatorname{div}\mathbf{v}=0$ and $\operatorname{div}\mathbf{H}=0$. Here $\mathbf{v}$, $\mathbf{E}$ and $\mathbf{H}$ are the velocity, electric-field, and magnetic-field vectors, while μ, ν, ρ and σ are the permeability, kinematic viscosity, density, and electrical conductivity of the fluid. Equation (1) is the momentum equation for the fluid, where the electromagnetic body force has been expressed with the aid of Ampère's Law. Equation (2) is Ohm's Law for the moving conductor, again using Ampère's Law. The assumptions made in deriving these equations are (i) steady flow and (ii) constancy of μ, ν and σ.

The case studied by Hartmann is sketched in figure 1; it is characterized by $\partial/\partial z=0=\partial/\partial x$ (except that a constant pressure gradient $\partial p/\partial x\equiv p_x$ may be present). It follows, with the aid of Faraday's Law and the boundary conditions indicated, that $\mathbf{v} = (v_x, 0, 0)$, $\mathbf{H} = (H_x, H_0, 0)$, and $\mathbf{E} = (0, 0, E_0)$, where H_0 and E_0 are given constants.

The equations are immediately reduced to

$$\frac{-\mu}{4\pi\rho} H_0 \frac{\partial H_x}{\partial y} + \frac{1}{\rho} p_x = \nu \frac{\partial^2 v_x}{\partial y^2}, \tag{3}$$

$$\frac{\partial}{\partial y}\left\{p + \frac{\mu}{8\pi} H_x^2\right\} = 0, \tag{4}$$

$$-\frac{\partial H_x}{\partial y} = 4\pi\sigma\{E_0 + \mu H_0 v_x\}. \tag{5}$$

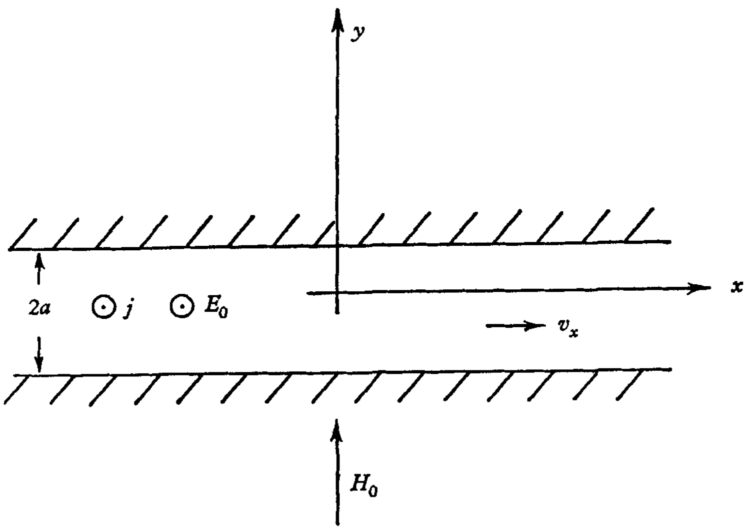

FIGURE 1. Sketch defining Hartmann flow.

Upon substitution of equation (5) into equation (3), an equation for v_x is obtained, which is easily integrated, yielding a familiar result (see Hartmann 1937; Resler & Sears 1958). For brevity we shall not write it out here in its exact form, but only exhibit it for the case of large Hartmann number

$$\mathrm{Ha} \equiv \mu H_0 a \sqrt{(\sigma/\rho\nu)},$$

in which the boundary-layer character is shown. For $y > 0$, it is

$$v_x \approx -\frac{p_x + \sigma\mu E_0 H_0}{\sigma\mu^2 H_0^2}\{1 - e^{-\mathrm{Ha}(1-y/a)}\}. \tag{6}$$

Near each wall there is a boundary layer of exponential profile, of thickness

$$\delta \propto \frac{a}{\mathrm{Ha}} = \frac{1}{\mu H_0}\sqrt{\left(\frac{\rho\nu}{\sigma}\right)}. \tag{7}$$

Across this layer the streamwise component of magnetic field, H_x, changes its value by an amount

$$4\pi\left\{\frac{p_x+\sigma\mu E_0 H_0}{\sigma\mu^2 H_0^2}\sqrt{(\rho\nu\sigma)}-\frac{p_x}{\mu H_0}\,\delta\right\}. \tag{8}$$

The difference in pressure across the layer is clear from equation (4).

These are the same as Stewartson's results (1960) except for our inclusion of the pressure gradient p_x; to recover his formulae, we need only put $p_x = 0$ and write U for the constant preceding the bracket in equation (6).

We shall refer to this type of boundary layer as a 'Hartmann layer'. It has the unusual properties of (i) constant thickness, and (ii) thickness proportional to the square root of the product of two diffusivities.

3. The boundary-layer equation

In this section we shall derive the appropriate equations for the description of the boundary layer in a general case of steady, two-dimensional, cross-fields flow at large Hartmann number. For simplicity and clarity of the argument, we shall suppose that the fluid is incompressible; the conclusions for compressible fluids are essentially the same.

Before undertaking this derivation, however, let us write out equations (1) and (2) in terms of dimensionless variables. The reference quantities employed to normalize $\mathbf{v}, \mathbf{H}$, $\mathbf{E}$, p, and the co-ordinates are respectively the undisturbed-flow quantities U and H_0, the combinations $\mu|\mathbf{U}\times\mathbf{H}_0|$ and $\frac{1}{2}\rho U^2$, and a reference length L. The equations are then

$$\mathbf{v}\,.\,\nabla\mathbf{v}-A^{-2}\mathbf{H}\,.\,\nabla\mathbf{H}+\operatorname{grad}P = Re^{-1}\nabla^2\mathbf{v} \tag{9}$$

and

$$\mathbf{E}+\mathbf{v}\times\mathbf{H} = Rm^{-1}\operatorname{curl}\mathbf{H}, \tag{10}$$

where

$$P \equiv p + A^{-2}\mathbf{H}^2,$$
$$A^2 \equiv 4\pi\rho U^2/\mu H_0^2,$$
$$Re \equiv UL/\nu,$$
$$Rm \equiv 4\pi\mu UL\sigma.$$

The divergence conditions are unchanged. Faraday's Law, in this two-dimensional steady case requires $\mathbf{E} = (0, 0, 1)$.

Equations (9) and (10) will now be written out in familiar boundary-layer co-ordinates. These are orthogonal curvilinear co-ordinates x and y, parallel and perpendicular, respectively, to the solid–fluid interface in the plane of the flow, and z normal to it. Specifically, y is the dimensionless distance from the interface and x the dimensionless distance measured along the interface (cf. Goldstein 1938). The differential operators of equations (9) and (10), when written out in these co-ordinates, naturally involve the dimensionless curvature $\kappa(x)$ of the surface. It seems important to carry through the derivation with this degree of generality because there are new terms and different orders of magnitude compared with the classical case of the Prandtl boundary layer.

Nevertheless, the process is greatly facilitated by reference to Goldstein (1938), where the equations of non-conducting fluids are written out in these co-ordinates. The form of equation (9) can be deduced immediately, since the only new terms are those of $\mathbf{H}.\nabla\mathbf{H}$, which are analogous to those of $\mathbf{v}.\nabla\mathbf{v}$. The complete equations are, for the two-dimensional case,

$$\frac{1}{1+\kappa y}v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+\frac{\kappa}{1+\kappa y}v_xv_y$$
$$-A^{-2}\left\{\frac{1}{1+\kappa y}H_x\frac{\partial H_x}{\partial x}+H_y\frac{\partial H_x}{\partial y}+\frac{\kappa}{1+\kappa y}H_xH_y\right\}+\frac{1}{1+\kappa y}\frac{\partial P}{\partial x}$$
$$=Re^{-1}\left\{\frac{1}{(1+\kappa y)^2}\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}-\frac{y}{(1+\kappa y)^3}\frac{d\kappa}{dx}\frac{\partial v_x}{\partial x}\right.$$
$$\left.+\frac{\kappa}{1+\kappa y}\frac{\partial v_x}{\partial y}-\frac{\kappa^2}{(1+\kappa y)^2}v_x+\frac{1}{(1+\kappa y)^3}\frac{d\kappa}{dx}v_y+\frac{2\kappa}{(1+\kappa y)^2}\frac{\partial v_y}{\partial x}\right\}, \quad (11)$$

and
$$\frac{1}{1+\kappa y}v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}-\frac{\kappa}{1+\kappa y}v_x^2$$
$$-A^{-2}\left\{\frac{1}{1+\kappa y}H_x\frac{\partial H_y}{\partial x}+Hy\frac{\partial H_y}{\partial y}-\frac{\kappa}{1+\kappa y}H_x^2\right\}+\frac{\partial P}{\partial y}$$
$$=Re^{-1}\left\{\frac{1}{(1+\kappa y)^2}\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}-\frac{y}{(1+\kappa y)^3}\frac{d\kappa}{dx}\frac{\partial v_y}{\partial x}\right.$$
$$\left.+\frac{\kappa}{1+\kappa y}\frac{\partial v_y}{\partial y}-\frac{\kappa^2}{(1+\kappa y)^2}v_y-\frac{1}{(1+\kappa y)^3}\frac{d\kappa}{dx}v_x-\frac{2\kappa}{(1+\kappa y)^2}\frac{\partial v_x}{\partial x}\right\}. \quad (12)$$

The form assumed by equation (10) can also be written out immediately, since Goldstein provides the curl operator (see pp. 102 and 119):

$$1+v_xH_y-v_yH_x=Rm^{-1}\left\{\frac{1}{1+\kappa y}\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-\frac{\kappa}{1+\kappa y}H_x\right\}. \quad (13)$$

Finally, both divergence-free conditions can be written as

$$\frac{\partial}{\partial x}\begin{pmatrix}v_x\\H_x\end{pmatrix}+\frac{\partial}{\partial y}\left\{(1+\kappa y)\begin{pmatrix}v_y\\H_y\end{pmatrix}\right\}=0. \quad (14)$$

The boundary-layer equations are now determined by collecting the leading terms of equation (11), (12) and (13) under the following assumptions:

(i) That the boundary layer is thin, of thickness δ, where $\delta\to 0$ as $\mathrm{Ha}\to\infty$. Taking our clue from the Hartmann layer, we assume explicitly that δ is of order $1/\mathrm{Ha}$, or, for fixed, arbitrary value of A,

$$\delta=O(Re\,Rm)^{-\frac{1}{2}}.$$

(ii) That the magnetic Prandtl number, Rm/Re, has a fixed, arbitrary value as $\mathrm{Ha}\to\infty$. In other words, both Re^{-1} and Rm^{-1} in the differential equations are of order δ.

(iii) That the curvature of the surface is not large; specifically that both $\delta\kappa$ and $\delta(d\kappa/dx)$ can be neglected in comparison with 1 as $\mathrm{Ha}\to\infty$.

Consider first the divergence conditions (equation (14)). For points within the boundary layer they state that both $\partial v_y/\partial y$ and $\partial H_y/\partial y$ are $O(1)$ at most. In view of the boundary condition $v_y(x,0)=0$, this means

$$v_y=O(\delta), \quad (15)$$

and that H_y varies only to order δ; i.e.

$$H_y = H_y(x) + O(\delta).\dagger \tag{16}$$

Now consider the leading terms of equation (13); they are $O(1)$; viz.

$$1 + v_x H_y(x) = -Rm^{-1}\,\partial H_x/\partial y. \tag{17}$$

The leading terms of equation (12) are seen to be $O(1)$; this equation therefore becomes a statement that

$$\partial P/\partial y = O(1) \tag{18}$$

(at most); i.e.

$$P = P(x) + O(\delta) \tag{19}$$

in the layer.

The orders of magnitude of terms in equation (11) can now be assessed; the leading terms are $O(\delta^{-1})$; viz.

$$-A^{-2}H_y(x)\,\partial H_x/\partial y = Re^{-1}\,\partial^2 v_x/\partial y^2. \tag{20}$$

Integrating with respect to y, we have

$$A^{-2}H_y(x)\,\{H_x(x,\delta) - H_x\} = Re^{-1}\,\partial v_x/\partial y, \tag{21}$$

where the arbitrary function of x has been evaluated at $y = \delta$, where (by definition) $\partial v_x/\partial y = 0$.

Since $\partial H_x/\partial y$ also vanishes at the outer edge of the layer, equation (17) includes the information that

$$1 + v_x(x,\delta)\,H_y(x) = 0, \tag{22}$$

which is nothing but the statement, in dimensionless two-dimensional form, that $\mathbf{E} + \mu\mathbf{v}\times\mathbf{H} = 0$ in an ideal conducting fluid. Thus equation (17) can be written

$$H_y(x)\,\{v_x(x,\delta) - v_x\} = Rm^{-1}\,\partial H_x/\partial y. \tag{23}$$

In both equations (21) and (23), the quantities in brackets are the *deficiencies* of H_x and v_y, respectively, relative to conditions just outside the boundary layer. Let these be called H_x' and v_x' for the time being; then equations (21) and (23) can be combined by cross-differentiation to read

$$\left[\frac{\partial^2}{\partial y^2} - \frac{Re\,Rm}{A^2}\{H_y(x)\}^2\right]\begin{pmatrix} H_x' \\ v_x' \end{pmatrix} = 0. \tag{24}$$

The solution is therefore

$$\begin{matrix} H_x' \\ v_x' \end{matrix} \propto \exp\{\pm\sqrt{(Re\,Rm)}\,A^{-1}H_y(x)\,y\}. \tag{25}$$

The combination $\sqrt{(Re\,Rm)}\,A^{-1}$ is recognized as the Hartmann number, Ha, of the flow.

† The symbol $H_y(x)$, of course, denotes the value of H_y at any value of y within the boundary layer; it will be used consistently to emphasize that this part of H_y (the largest part) is a function of x only. The analogous notation is used in equation (19). For brevity the independent variables are not written out in general for the various functions when they are dependent upon both x and y.

The constant of proportionality in v'_x is determined immediately by the no-slip boundary condition, which is $v'_x = v_x(x, \delta)$ at $y = 0$; thus (for $y > 0$),

$$v_x = v_x(x, \delta)\,[1 - \exp\{-\mathrm{Ha}\,H_y(x)\,y\}]. \tag{26}$$

The constant of porportionality in H'_x can be determined by comparing $-\partial H'_x/\partial y$ with $\partial H_x/\partial y$ in equation (23), namely $Rm\,H_y(x)\,v'_x$. The result is

$$H_x = H_x(x, \delta) - A\sqrt{(Rm/Re)}\,v_x(x, \delta)\exp[-\mathrm{Ha}\,H_y(x)\,y]. \tag{27}$$

Equations (26) and (27) do not exhaust the conclusions that can be drawn from the first-order (in δ) boundary-layer theory, of course, but they are sufficient for our subsequent discussion.

4. Discussion of equations

Surprisingly, it has been possible to integrate the equations of the two-dimensional crossed-fields boundary layer in general, making only very reasonable assumptions about the curvature of the interface. The results, equations (26) and (27), state that the boundary layer is always a Hartmann-type layer and is described by equations analogous to equations (6)–(8) with Ha replaced by the *local* Hartmann number based on the normal component of **H** and with U replaced by the local flow speed outside the layer. Although we cannot agree with Clauser that 'no boundary layer can exist', we must agree that it is not a conventional layer. Its profile is always exponential and its thickness depends completely on local quantities.†

Essentially, the reason for this is that the viscous shear and the body force are in equilibrium; the boundary layer does not pass along a momentum deficiency to the stations downstream. For example, let us calculate the local skin friction. From equation (26), it is (per unit length of the wall)

$$\rho\nu(\partial v_x/\partial y)_{y=0}\,U/L = \rho\nu v_x(x, \delta)\,\mathrm{Ha}\,H_y(x)\,U/L,$$

or the local skin-friction coefficient is

$$2\{\mathrm{Ha}\,H_y(x)\}/\{Re\,v_x(x, \delta)\}.$$

The total body force on the boundary layer (per unit length of the wall) is the product of $H_0 H_y(x)$ and $1/4\pi$ times the jump in $H_x H_0$ across the layer, from equation (27):

$$\frac{1}{4\pi}H_0^2 H_y(x)\,A\sqrt{(Rm/Re)}\,v_x(x, \delta),$$

which is exactly the same.

Let us consider the process of solving flow problems in this category. As is characteristic of boundary-layer flows, the external flow solution for the limit $\mathrm{Ha} = \infty$ must be solved first. This flow involves a vortex sheet at the interface, as usual, but this phenomenon has no effect on the flow.

† Prof. I. Imai points out the analogy between this boundary layer and the boundary layer with suction. In Schlichting (1960), pp. 271, 272, the case of the flat plate is analysed. It does not seem to have been noted in the literature that, as the present analysis shows, the results apply locally to more general flows where U_∞ and v_0 (in Schlichting's notation) are variable.

However, equation (27) states that this external-flow solution *also involves a current sheet* at the interface, of strength

$$A\sqrt{(Rm/Re)}\, v_x(x,\delta).$$

In other words, the boundary condition of continuity of H_x at the interface, which is often employed (Sears & Resler 1959) is correct only for magnetic Prandtl number ($Rm/Re \equiv Pr_m$) equal to zero (cf. Stewartson 1960).

For other values of Pr_m, the more complex boundary condition, involving as it does one of the results of the calculation, is appropriate. This complication, however, seems to render more difficult a subject that is already practically unsolvable: there are no known general methods of solving crossed-fields flow problems, even for ideal inviscid conductors. Note that even the flat-plate problem is intractable in this respect. The current-sheet strength is $O(Pr_m)^{\frac{1}{2}}$ which is $O(1)$; the crossed-fields flow of an ideal inviscid conductor past such a current sheet is not known.

Before proceeding to seek a solvable case, we interpolate here some remarks concerning the approach to two limiting cases, namely (i) non-magnetic flow ($H_0 \to 0$), and (ii) aligned-fields flow ($\mathbf{U} \times \mathbf{H}_0 \to 0$). In both limits the boundary layer should change its character dramatically, increasing in thickness to $O(Rm^{-\frac{1}{2}})$ or $O(Re^{-\frac{1}{2}})$, and losing its simple, local-Hartmann nature. These limits are not within the scope of the present theory, however. Non-magnetic flow is obviously eliminated by the assumption of large Hartmann number. That the aligned-fields limit is also excluded can be seen in equations (24)–(27), where $H_y(x)$ vanishes in the limit. Then H_x and v_x are essentially constant through the thin layer considered here, which means that the boundary layer is an order of magnitude thicker.

5. Flat-plate flow at small Pr_m

We shall now show that approximate results can be obtained, to order $Pr_m^{\frac{1}{2}}$, for small Pr_m. Consider the case of flat-plate flow with perpendicular fields; since the only disturbance to the fields in the limit $\mathrm{Ha} = \infty$ is due to the current s heet, it is permissible to assume a small-perturbation solution of the form

$$\left.\begin{aligned} \mathbf{v} &= (1+u', v', 0), \\ \mathbf{H} &= (h_x, 1+h_y, 0), \end{aligned}\right\} \tag{28}$$

where u', v', h_x, and h_y are all small compared with 1 and x, y are rectangular Cartesian co-ordinates aligned with the free stream (and the plate) and the imposed magnetic field, respectively.

But this is exactly what was assumed in our study of crossed-fields flow past thin cylinders (Sears & Resler 1959), and the results can be carried over immediately. In particular (from equations (48) to (55) of Sears & Resler 1959),

$$u' = \frac{A}{1+A^2}\{F(x-Ay) - G(x+Ay)\} + \frac{\partial\phi}{\partial x}, \tag{29}$$

$$v' = \frac{1}{1+A^2}\{F(x-Ay) + G(x+Ay)\} + \frac{\partial\phi}{\partial y}, \tag{30}$$

$$h_x = \frac{-A^2}{1+A^2}\{F(x-Ay)+G(x+Ay)\}+\frac{\partial\phi}{\partial y}, \tag{31}$$

$$h_y = -u', \tag{32}$$

where F and G are arbitrary functions, identical with F_1 and G_1 of Sears & Resler (1959) but made dimensionless by dividing by U, and ϕ is a harmonic function, identical with ϕ_1 but divided by UL.

Our knowledge of the mechanism of Alfvén-wave propagation permits us to put F and G, respectively, equal to zero in certain regions. Let the flow field be subdivided as in figure 2 (*c*), then:

In I and IV: $F = 0 = G$. There are no discontinuities; the flow is wave-free, current-free, and irrotational.

In II: $G = 0$. A boundary condition is $v'(x, 0+) = 0$; thus,

$$(1/1+A^2)\,F(x)+\phi_y(x,0+) = 0, \tag{33}$$

where $\phi_y(x,y)$ denotes $\partial\phi/\partial y$.

In *III*: $F = 0$.

$$(1/1+A^2)\,G(x)+\phi_y(x,0-) = 0. \tag{34}$$

This flow field must be symmetrical about the x-axis, i.e. invariant to a reflexion of y in $-y$, since the direction assigned to $\mathbf{H}$ is purely conventional. Thus u' and h_x must be even in y, and v' and h_y odd; this requires that

$$F = -G \quad \text{and} \quad \phi(x,y) = \phi(x,-y). \tag{35}$$

In the light of equations (35), equations (33) and (34) state that the jump in ϕ_y across the plate, say $[\phi_y(x)]$, is

$$[\phi_y(x)] \equiv \phi_y(x,0+)-\phi_y(x,0-)$$
$$= (-2/1+A^2)\,F(x). \tag{36}$$

We now introduce the jump condition on h_x, from equation (27). Since the jump in h_x across the plate-plus-boundary layers is twice the jump across the boundary layer, this statement is

$$[h_x(x)] \equiv h_x(x,0+)-h_x(x,0-) \tag{37}$$
$$= 2A\sqrt{(Pr_m)}\,v_x(x,0+)$$
$$= 2A\sqrt{(Pr_m)}\,\{1+O(\sqrt{Pr_m})\}, \tag{38}$$

which is just $2A\sqrt{Pr_m}$ in our first-order (in $\sqrt{Pr_m}$) theory. Let us call this small quantity 2ϵ.

According to equation (31), then

$$2\epsilon = -(A^2/1+A^2)\{F(x)-G(x)\}+[\phi_y(x)] \tag{39}$$

$$= -\frac{2A^2}{1+A^2}F(x)-\frac{2}{1+A^2}F(x) = -2F(x). \tag{40}$$

Thus
$$F(x) = \text{const.} = -\epsilon = -G(x) \tag{41}$$
in this case.

Furthermore, the jump $[\phi_y(x)]$ is now seen to be a constant, namely $2\epsilon/(1+A^2)$. That there is no jump in ϕ_x follows immediately from the symmetry, equation (35); thus $\phi(x,y)$ represents the flow due to a source sheet at most, and not a

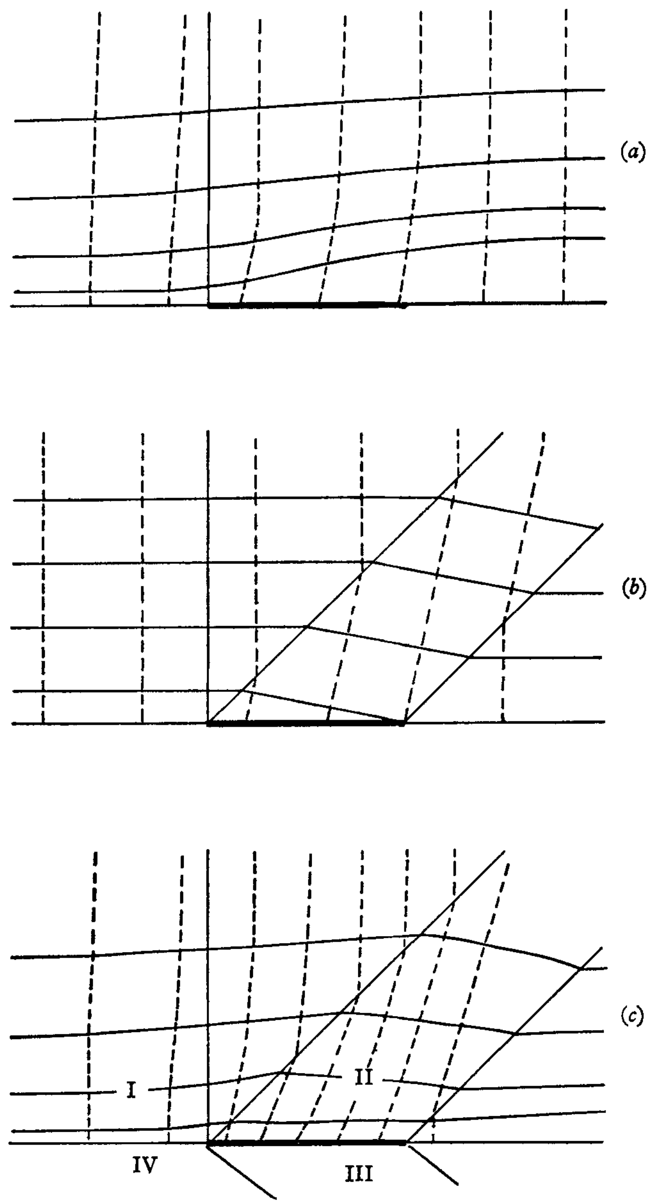

FIGURE 2. Sketches showing small-perturbation flow past flat plate in the limit Ha $= \infty$. The parameter ϵ, defined as $2A\sqrt{Pr_m}$, has been taken equal to $\frac{1}{8}$ and $A = 1$. Sketch (a) shows the harmonic part of the solution and (b) the rotational part. Sketch (c) is the combined flow. In each case streamlines are solid and magnetic lines of force dashed. The direction of flow is from left to right.

vortex sheet. The flow field of this source sheet, having the constant strength $2\epsilon/(1+A^2)$, is given by

$$\phi_x(x,y) = \frac{-\epsilon}{2\pi}\frac{1}{1+A^2}\ln\frac{(x-1)^2+y^2}{x^2+y^2}, \tag{42}$$

$$\phi_y(x,y) = \frac{-\epsilon}{\pi}\frac{1}{1+A^2}\left\{\tan^{-1}\left(\frac{x-1}{y}\right)-\tan^{-1}\left(\frac{x}{y}\right)\right\}. \tag{43}$$

This is the flow field of incompressible irrotational flow past a slender wedge, as sketched in figure 2 (*a*). The rotational part of the total flow field is given by

$$\begin{aligned} F(x-Ay) &= -\epsilon \quad \text{for} \quad y>0 \quad \text{and} \quad 0<x-Ay<1, \\ &= 0 \quad \text{elsewhere}, \\ G(x+Ay) &= \epsilon \quad \text{for} \quad y<0 \quad \text{and} \quad 0<x+Ay<1, \\ &= 0, \quad \text{elsewhere}, \end{aligned}$$

which is analogous to, but not identical with, supersonic flow past a reversed wedge, as sketched in figure 2 (*b*). The streamlines of the superimposed flow are sketched in figure 2 (*c*).

In each case the corresponding configuration of magnetic lines of force is also sketched.

6. Flow past thin cylinders at small Pr_m

The results of our previous calculations (Sears & Resler 1959) for thin cylinders (airfoils) can now be generalized to the case of non-zero but small Pr_m provided that only first-order effects in $\sqrt{Pr_m}$ and first-order effects in body thickness are considered. Equations (29)–(32) apply once again, and so does equation (38), which establishes the magnitude of the boundary-layer effect; namely $A\sqrt{Pr_m}$ for any thin cylinder.

Thus, all four of the perturbation vector components are simply linear combinations of the results obtained in Sears & Resler and those obtained in § 5 of the present paper.

The lift and moment on the cylinder are, of course, unaffected by the boundary-layer effect, by virtue of its symmetry. The drag, on the other hand, involves two new terms related to viscosity. The first is the skin-friction drag; according to § 4 the value of this term, to first order in Pr_m, is

$$2\rho U\nu\,\mathrm{Ha}, \quad \text{or} \quad 2\rho U^2 L\sqrt{(Pr_m)}/A$$

per unit length of the cylinder.

The second drag term is of higher order. It results from the non-uniform pressure perturbation due to the boundary-layer effect calculated above, if the body has incidence or camber. Its order of magnitude is given by the product of $\sqrt{Pr_m}$ and the body-thickness parameter.†

† The drag mentioned in Sears & Resler (1959, p. 269), for the case $Pr_m = 0$, is, of course, proportional to the body-thickness parameter squared.

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